MATH 303 – Measure Theory
Homework 3

Please upload a pdf of your solutions by 23:59 on Monday, October 13. The assignment will be graded out of 10 points, taking into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

This problem makes use of the Lebesgue measure on the real line, which we will construct (along with a much wider family of measures) in future lectures. All that you need to know about the Lebesgue measure (which we will denote by λ) for this problem is that λ is a measure defined on all Borel subsets of $\mathbb{R}$ such that $\lambda((a,b)) = b - a$ for every interval $(a,b) \subseteq \mathbb{R}$, and λ is translation-invariant in the sense that $\lambda(B+t) = \lambda(B)$ for every Borel set $B \subseteq \mathbb{R}$ and every $t \in \mathbb{R}$.

Problem 1.

(a) Show that for every irrational number $x \in \mathbb{R}$, there are infinitely many rational numbers $\frac{p}{q}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

(Hint: This part of the problem does not require any measure theory.)

(b) For $c > 2$, let A_c be the set of real numbers $x \in \mathbb{R}$ for which there are infinitely many rational numbers $\frac{p}{q}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^c}.$$

Show that A_c is a Borel set.

(c) Prove that if $c > 2$, then $\lambda(A_c) = 0$.

Solution: (a) Consider the sequence $x, 2x, 3x, \ldots$. Fix $N \in \mathbb{N}$ and divide the real line into N sets $I_j = \mathbb{Z} + \left(\frac{j}{N}, \frac{j+1}{N}\right)$ for $j \in \{0, 1, \ldots, N-1\}$. By the pigeonhole principle, there must be two terms ax and bx with $1 \leq a < b \leq N+1$ that belong to the same set I_j for some j. (Note that nx never takes a value on the boundary of I_j, since x is irrational.) But then taking $q = (b-a)$, we have $q \in \{1, \ldots, N\}$ and $qx \in \mathbb{Z} + \left(-\frac{1}{N}, \frac{1}{N}\right)$. Let $p \in \mathbb{Z}$ such that $|qx - p| < \frac{1}{N}$. Dividing both sides by q, we have

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN} \leq \frac{1}{q^2}.$$

Using the first inequality $\left| x - \frac{p}{q} \right| < \frac{1}{qN}$ and irrationality of x (so that $x \neq \frac{p}{q}$), we see that each rational number $\frac{p}{q}$ can only appear for finitely many $N \in \mathbb{N}$. Thus, there are infinitely many different rational numbers $\frac{p}{q}$ satisfying the desired inequality.

(b) Note that if $\frac{p}{q}$ is a rational number that is not in reduced terms, say $\frac{p}{q} = \frac{ap'}{aq'}$ for some $p' \in \mathbb{Z}$, $q' \in \mathbb{N}$, $a \geq 2$, and

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^c},$$

then

$$\left|x - \frac{p'}{q'}\right| = \left|x - \frac{p}{q}\right| < \frac{1}{q^c} = \frac{1}{a^c (q')^c} < \frac{1}{(q')^c}.$$

Therefore, it suffices to consider only reduced fractions.

For each $q \in \mathbb{N}$, let

$$A_{c,q} = \bigcup_{\substack{p \in \mathbb{Z}, \\ \gcd(p,q)=1}} \left(\frac{p}{q} - \frac{1}{q^c}, \frac{p}{q} + \frac{1}{q^c} \right).$$

Then $A_{c,q}$ is an open set and

$$A_c = \{x \in \mathbb{R} : x \in A_{c,q} \text{ for infinitely many } q \in \mathbb{N}\} = \bigcap_{Q \in \mathbb{N}} \bigcup_{q \geq Q} A_{c,q}.$$

Thus, A_c is a countable intersection of open sets (a G_δ-set), so it is Borel.

(c) Let $A_c' = A_c \cap [0,1)$, and observe that $A_c = \bigcup_{n \in \mathbb{Z}} (A_c' + n)$. Indeed, given $x \in \mathbb{R}$, we may write $x = n + x'$ for some $n \in \mathbb{Z}$ and $x' \in [0,1)$, and rational approximations of x are in 1:1 correspondence with approximations of x' with the same denominator:

$$\left|x - \frac{p}{q}\right| = \left|n + x' - \frac{p}{q}\right| = \left|x' - \frac{(p - nq)}{q}\right|.$$

Since λ is countably additive and $\lambda(A_c' + n) = \lambda(A_c')$, it suffices to show $\lambda(A_c') = 0$. We can write

$$A_c' = \{x \in \mathbb{R} : x \in A_{c,q}' \text{ for infinitely many } q \in \mathbb{N}\} = \bigcap_{Q \in \mathbb{N}} \bigcup_{q \geq Q} A_{c,q}',$$

where $A_{c,q}' = A_{c,q} \cap [0,1)$. Now, for each $q \in \mathbb{N}$,

$$\lambda(A_{c,q}') = \lambda\left(\left[0, \frac{1}{q^c}\right)\right) + \sum_{p=1}^{q-1} \lambda\left(\left(\frac{p}{q} - \frac{1}{q^c}, \frac{p}{q} + \frac{1}{q^c}\right)\right) + \lambda\left(\left(1 - \frac{1}{q^c}, 1\right]\right)$$
$$= \frac{1}{q^c} + \frac{2}{q^c} \cdot (q-1) + \frac{1}{q^c} = \frac{2}{q^{c-1}}.$$

Since $c - 1 > 1$, we conclude that

$$\sum_{q=1}^{\infty} \lambda(A_{c,q}') < \infty,$$

so $\lambda(A_c') = 0$ by the Borel–Cantelli lemma.